Skip the Square Root of $n$: A New Secretary Problem

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Abstract

We present an extension of the secretary problem in which the DM sequentially observes up to $n$ applicants whose actual values $X_1, X_2, \ldots, X_n$ are drawn i.i.d. from a uniform distribution on $[0, 1]$. The DM must select exactly one applicant, cannot recall released applicants, and receives a payoff of $x_t$, the realization of $X_t$, for selecting the $t$th applicant. For each encountered applicant, the DM only learns whether the applicant is the best so far. We prove that the optimal policy dictates skipping the first $n^{2/3} - 1$ applicants, and then selecting the next encountered applicant with rank 1.

Key words: optimal stopping, sequential search, secretary problem

1 Introduction

Suppose a decision maker (DM) observes a sequence of up to $n$ applicants whose values are random variables $X_1, X_2, \ldots, X_n$ drawn i.i.d. from a uniform distribution on $[0, 1]$. As in the standard secretary problem, the DM has two choices for each applicant: accept or reject. The DM’s payoff for selecting an applicant with $X_t = x_t$ is itself $x_t$, the realization of $X_t$. Once an applicant is selected the problem terminates; if reached, the $n$th applicant must be accepted; and, once rejected, an applicant cannot be recalled. Importantly,

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however, at each stage $t$ the DM only observes an indicator of $X_t$, where $I_t = 1$ if and only if $x_t = \max \{x_1, x_2, \ldots, x_t\}$; otherwise, $I_t = 0$. In other words, the DM only learns whether each observed applicant is the best so far. Her objective is to maximize her expected payoff.

Thus, the current problem is quite similar to the standard secretary problem (for reviews of secretary problems see Ferguson, 1989; Samuels, 1991). The only difference is that in our problem the DM’s payoff is equal to the selected applicant’s underlying “true” value, whereas in the classical secretary problem the DM earns a payoff of 1 if she selects the best overall applicant and earns nothing otherwise. Our motivation for this problem is the intuition that in some sequential search situations, DMs might pay close attention to rank-based information, but, ultimately, derive utility from the cardinal (true) value of the selected observation and not from its rank. Consider a trader (hirer) who wants to sell an asset when its price (applicant) is at its maximum during some period of time $[t_{\min}, t_{\max}]$. Though the price ranks are salient in deciding when to sell, presumably she will derive utility that is strictly increasing in cardinal selling price. The nothing-but-the-best payoff scheme of the classical secretary problem fails to capture this.

Supposing our trader does make her selling decisions at each point in time $t$ solely on the basis of the rank of the current price with respect to the previous prices, how can she maximize her expected selling price? And how well might she do given that her earnings will ultimately be based on the price at which the asset is sold and not on the rank of the asset price?

Next, we present the optimal stopping policy for our problem and show some of its basic properties. We conclude by suggesting that this problem is worthy of empirical study with actual DMs.

2 The Optimal Policy

Since the applicant’s values are i.i.d. draws from a uniform distribution on $[0, 1]$, the expected value of the $t$th applicant given that $x_t = \max \{x_1, x_2, \ldots, x_t\}$ is given by

\[ \frac{1}{t}. \]

\[ \text{Obviously, asset prices are not sampled i.i.d. from a uniform distribution. We are exercising license simply for illustration.} \]
\[
E_t = E(X_t | I_t = 1) = \frac{\int_0^1 F(x)^{t-1} f(x) dx}{\int_0^1 F(x)^{t-1} f(x) dx}
= \frac{\int_0^1 x^{t-1} dx}{\int_0^1 x^{t-1} dx}
= \frac{t}{t+1}.
\]

(1)

Note that this is a standard result on the \(n\)th order statistic for the uniform distribution.

By the Principle of Optimality, since \(dE_t/dt > 0\), if it is optimal to select an applicant with \(I_t = 1\), then it is optimal to select an applicant with \(I_{t+k} = 1\), \(\forall k \geq 0\). Obviously, for \(1 \leq t < n\), it is never optimal to select an applicant for which \(I_t = 0\). Let us, therefore, define with smallest \(t\) at which it is optimal to select an applicant with \(I_t = 1\) for a problem with \(n\) applicants as \(c^*_n\). We refer to \(c^*_n\) as the optimal cutoff.

It may help to begin by expressing the value of a policy by means of a recurrence. Denote the expected payoff of the selected applicant—the value—when applying some (possibly non-optimal) cutoff \(1 \leq c \leq n\) from stage \(c\) to stage \(n\) by \(V_n(c)\). One can easily obtain \(V_n(c)\) by working backward from stage \(n\) to stage \(c\). Since the DM must stop at stage \(n\), if reached, \(V_n(n) = \frac{1}{2}\). Then, for \(t = n-1\) to \(c\),

\[
V_n(t) = \left(\frac{1}{t}\right) E_t + \left(1 - \frac{1}{t}\right) V_n(t+1)
= \left(\frac{1}{t+1}\right) + \left(\frac{t-1}{t}\right) V_n(t+1).
\]

(2)

For \(1 \leq t < c\), \(V_n(t) = V_n(c)\). Evoking the Principle of Optimality, an optimal DM will stop at stage \(t\) when \(I_t = 1\) only if \(E_t \geq V_n(t+1)\), that is, if the expected payoff for stopping is not less than the expected payoff for continuing. Optimal policies can, therefore, be obtained quite easily by dynamic programming. Recursively applying Eq. 2, one solves

\[
c^*_n = \arg \max\limits_{c \in \{1, 2, \ldots, n\}} V_n(c).
\]

(3)

The value of the optimal policy for a problem of size \(n\) is \(V^*_n \equiv V_n(c^*_n)\).

Fortunately, Eq. 3 can be solved without resorting to dynamic programming. Optimal policies and their values can easily be obtained from the proof of the
following proposition.

**Proposition 1** \( c_n^* \in \{\lfloor n^{\frac{1}{2}} \rfloor, \lceil n^{\frac{1}{2}} \rceil \} \).

**Proof**

Given a problem of size \( n \), the expected payoff for some arbitrary cutoff \( 1 \leq c \leq n \) can be obtained by

\[
V_n(c) = \left( \frac{1}{t+1} \right) + \sum_{t=c+1}^{n-1} \left[ \prod_{s=c}^{t-1} \left( \frac{s-1}{s} \right) \right] \left( \frac{1}{t+1} \right) + \left[ \prod_{s=c}^{n-1} \left( \frac{s-1}{s} \right) \right] \frac{1}{2}
\]

\[
= \sum_{t=c}^{n-1} \left( \frac{c-1}{t-1} \right) \left( \frac{1}{t+1} \right) + \left( \frac{c-1}{n-1} \right) \frac{1}{2}
\]

\[
= \frac{1}{c+1} + (c-1) \left( \frac{1}{c(c+2)} + \cdots + \frac{1}{(n-2)n} \right) + \left( \frac{c-1}{n-1} \right) \frac{1}{2}
\]

\[
= \frac{2cn - c^2 + c - n}{2cn}.
\]  

(4)

The last term on the right-hand side corresponds to the event that the DM makes it to the last applicant and is forced to accept him.

The first derivative of the value function is therefore

\[
\frac{\partial V}{\partial c} = -\frac{c^2 + n}{2c^2 n}.
\]  

(5)

Noting that \( \frac{\partial^2 V}{\partial c^2} < 0 \) for all permissible values of \( c \), setting \( \frac{\partial V}{\partial c} = 0 \) and solving for \( c \), we find that \( V \) reaches its maximum at \( c = n^{\frac{1}{2}} \). Since \( V \) is convex in \( c \), \( c_n^* \), which is integer-valued, must be in \( \{\lfloor n^{\frac{1}{2}} \rfloor, \lceil n^{\frac{1}{2}} \rceil \} \). Thus, the proposition is proved. \( \square \)

For \( n \) that are perfect squares, \( c_n^* = n^{\frac{1}{2}} \). When this condition does not hold, \( c_n^* \) can be found by evaluating \( V_n(c) \) at the points given by Proposition 1.

**Proposition 2** \( V_n^* \) is increasing in \( n \).

**Proof**

Given Eq. 4, we can find that
\[
\frac{\partial V}{\partial n}\bigg|_{c=n^{\frac{1}{2}}} = \frac{n - n^{\frac{1}{2}}}{2n^{\frac{3}{2}}},
\]

which is positive for all \(n > 1\). \(\square\)

Relatedly, it is easy to verify that

\[
\lim_{n \to \infty} V^*_n = 1.
\]

The values for handful of \(n\), along with their optimal cutoffs, are displayed in Table 1.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Optimal cutoffs and values for several values of (n).</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>(c^*_n)</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
</tr>
<tr>
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<tr>
<td>100000</td>
<td>316</td>
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</tbody>
</table>

3 Conclusion

The secretary problem has received considerable attention by statisticians and applied mathematicians. One reason for this is the problem’s surprising optimal policy. Under it, the DM skips the first \(s^*_n - 1\) applicants and then takes the next candidate (i.e., an applicant with relative rank 1). What is surprising is that \(s^*_n \to e^{-1}n\) as \(n \to \infty\), and that in the limit the best overall applicant is selected with probability \(e^{-1}\). A proof of this can be found in Gilbert and Mosteller (1966).

The problem introduced in the current note has a similar, to us, non-intuitive solution: skip the first \(c^*_n - 1\) applicants and then take the next applicant with rank 1, where \(c^*_n = n^{\frac{1}{2}}\). Aside from the formal derivation of this result, we do not yet have an dinner table explanation that expresses why the result obtains.

\cite{footnote} Technically, as shown in Proposition 1, \(c^*_n \in \{\lfloor n^{\frac{1}{2}}\rfloor, \lceil n^{\frac{1}{2}}\rceil\}\). For simplicity of presentation, with only a slight loss of accuracy, we write \(c^*_n = n^{\frac{1}{2}}\), which holds exactly when \(n\) is a perfect square.
Seale and Rapoport (1997) found that subjects in an experimental study of the classical secretary problem tended to terminate their search earlier than is dictated by the application of the optimal policy. Studies of a number of related problems have tended to find the same general result, namely that subjects do not search enough (e.g., Bearden et al., 2004; Seale and Rapoport, 2000; Zwick et al., 2003). For most values of \( n \), the optimal cutoff for the problem proposed in this note appear considerably earlier than the corresponding cutoffs for the classical secretary problem. Given the strictness of the payoff function for the classical problem, which returns a positive payoff if and only if the best of the \( n \) applicants is selected, one wonders whether the results obtained in these experimental studies are an artifact of the problem’s unusual payoff scheme. Our aforementioned trader may try to dump her asset when its price is its greatest during some interval, but it seems unlikely that her utility for selling at some prices slightly below the maximum would be zero. Compared to the classical secretary problem, it seems to us that the payoff scheme presented in this note is more natural. In the future, we intend to test the generality of the early stopping result using this new variation on the secretary theme.

References


